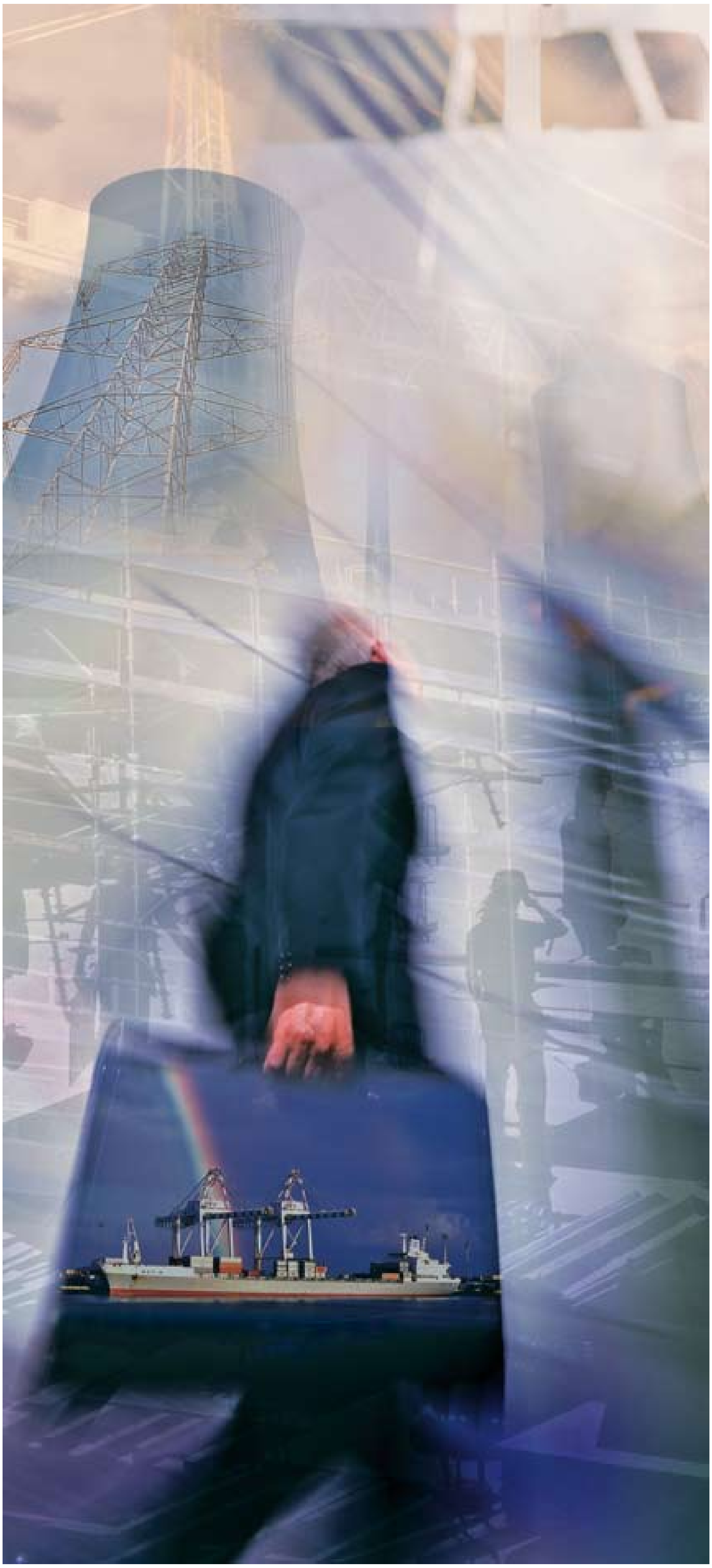

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A Geometric Comparison of the Delta and Fieller Confidence Intervals

Joe Hirschberg and Jenny Lye[1]

Abstract:

The comparison of the Delta and Fieller confidence intervals for the ratio of normally distributed means has long been of interest. The contribution of this paper is the construction of a common geometric representation of both the Delta and the Fieller intervals defined by two related optimization problems which are subject to a common constraint. The diagrammatic solution to these problems can be used to examine how earlier comparisons based on alternate analytic relationships and simulations have resulted in the particular conclusions they report. We find that along with the univariate statistics for the denominator and numerator, the agreement of signs of their correlation and ratio is crucial in determining the degree to which the intervals based on the Fieller and the Delta coincide.

Key words: constrained optimization, cost-effectiveness ratio, dynamic regression, estimated elasticities, bioassay

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1. Introduction

A statistic defined as the ratio of two normally distributed random variables is often encountered in applied work. The Delta method has been nominated as the most common technique for drawing inferences for such nonlinear combinations. The primary alternative for the computation of the confidence intervals of ratios is the Fieller method (or theorem) (1932, 1944, and 1954) which is derived from the properties of a ratio of bivariate normally distributed random variables (see Marsaglia (1965) and Hinkley (1969) for a detailed discussion of these cases and Zerbe (1978) for an application to the general linear model).

The wide spread choice of the Delta method may be due to the perception that Fieller's method is non-intuitive since it does not use the familiar construction of the confidence interval which rely on estimates for the mean and variance of the statistic (the ratio). Furthermore, besides being asymmetric the Fieller (1-α)100% confidence interval for $\alpha > 0$ may be unbounded on one or both sides, for example Raftery and Schweder (1993) reject the use of the Fieller interval on these grounds. Also it is more difficult to apply because it may require the application of special programming. And lastly, although some generalization of the Fieller method can be made to the class of likelihood profile tests it is a "one-off" type test that is only applied to inferences for ratios and thus is found in few textbooks.

Given these issues there has been an ongoing debate in a number of disciplines as to the relative merits of the Delta and the Fieller methods. Because the nature of the distribution of the variables used in the ratios found in these applications varies, a number of researchers have asked the questions: *Why do the Fieller and Delta differ in their inferences?* and *When can one rely on the Delta given the difficulties in the implementation of the Fieller?* These previous comparisons have either relied on an oversimplified relationship between the analytic form of the Fieller and the Delta or they have resorted to the use of simulations for particular cases. Here we contribute to this literature by demonstrating how the construction of a common geometric representation for both methods that can account for the variation in these five parameters that defines each interval.

In order to use a geometric exposition we first cast both the Fieller and Delta methods as solutions to constrained optimization problems that share a common constraint. By shifting the comparison away from algebraic and numeric comparisons we attempt to make the comparison of these two methods more intuitive. Although various diagrammatic approaches for the Fieller have appeared (See Fieller 1932, 1954, Creasy 1954, Guiard 1989, von Luxburg and Franz 2004, Hirschberg & Lye 2007), a major contribution of this paper is the construction of an equivalent diagram for the Delta approximation of the bounds for the ratio.

This paper proceeds as follows: First we derive a geometric representation of the Delta method for the estimation of the confidence intervals for a ratio of normally distributed random variables. Then we review the geometric representation of the Fieller method. By combining both geometric solutions into one diagram we then examine how the geometric solutions are influenced by the joint distribution of the numerator and denominator. Finally we interpret the conclusions from some past analysis in light of the geometric approach.

2. A geometric representation of the Delta confidence interval for a ratio of parameter estimates.

To generate an estimate of the variance of the function the Delta method applies a first order Taylor series expansion to linearize a nonlinear relationship. A confidence interval is then formed by assuming that the variables in this function are normally distributed. In the case we consider here we define a ratio of two parameters β_1 and β_2 as $\psi(\beta_1, \beta_2) = \beta_1/\beta_2$ which are estimated by b_1 and b_2 that are normally distributed according to:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \sim N\left\{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}\right\} \quad (1)$$

The first order Taylor series approximation of ψ is defined as $\tilde{\psi}(\beta_1, \beta_2)$ and is given by:

$$\tilde{\psi}(\beta_1,\beta_2)=\psi(\beta_{1,0},\beta_{2,0})+\begin{bmatrix}\frac{\partial\psi(\beta_{1,0},\beta_{2,0})}{\partial\beta_1} & \frac{\partial\psi(\beta_{1,0},\beta_{2,0})}{\partial\beta_2}\end{bmatrix}\begin{bmatrix}\beta_1-\beta_{1,0}\\ \beta_2-\beta_{2,0}\end{bmatrix} \tag{2}$$

Evaluating the approximation ($\tilde{\psi}$) at the estimates of β_1,β_2 defined by $\beta_{1,0}=b_1$, $\beta_{2,0}=b_2$ and $\psi(b_1,b_2)=\hat{\psi}$, the approximation is found to be a linear combination of the parameters:

$$\begin{aligned}\tilde{\psi}&=\hat{\psi}+\left(\tfrac{1}{b_2}\right)(\beta_1-b_1)-\left(\tfrac{b_1}{b_2^2}\right)(\beta_2-b_2)\\ &=\hat{\psi}+\left(\tfrac{1}{b_2}\right)\beta_1-\left(\tfrac{\hat{\psi}}{b_2}\right)\beta_2\end{aligned} \tag{3}$$

Thus $\tilde{\psi}$ is distributed according to:

$$\tilde{\psi}\sim N\left\{\psi,\left(\left(\tfrac{1}{b_2}\right)^2\sigma_{11}+\left(\tfrac{\hat{\psi}}{b_2}\right)^2\sigma_{22}-2\left(\tfrac{\hat{\psi}}{b_2^2}\right)\sigma_{12}\right)\right\} \tag{4}$$

And we can construct a 100(1-α)% confidence interval for $\tilde{\psi}$ based on this expression as:

$$\tilde{\psi}_{1-\alpha/2},\tilde{\psi}_{\alpha/2}=\hat{\psi}\pm z\sqrt{\left(\tfrac{1}{b_2}\right)^2\sigma_{11}+\left(\tfrac{\hat{\psi}}{b_2}\right)^2\sigma_{22}-2\left(\tfrac{\hat{\psi}}{b_2^2}\right)\sigma_{12}} \tag{5}$$

where z is the value of the standard normal distribution with probability (1-α/2).

Equivalently, it can be shown that the 100(1-α)% confidence interval for a linear combination of a vector of normally distributed random variables is the solution to the constrained optimization problem as proposed by Durand (1954) and Scheffé (1959 appendix III)

$$\mathcal{L}=\mathbf{a}'\boldsymbol{\beta}+\lambda\left((\boldsymbol{\beta}-\mathbf{B})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}-\mathbf{B})-z^2\right) \tag{6}$$

Where $\mathbf{B}_{k\times1}\sim N(\boldsymbol{\beta}_{k\times1},\boldsymbol{\Sigma}_{k\times k})$, λ is the Lagrange multiplier, the linear combination is defined as $\theta=\mathbf{a}'\boldsymbol{\beta}$ and $\mathbf{a}$ is a $k\times1$ constant vector (see Appendix A for a proof). The optimization defined by (6) has the advantage that one can determine a geometric solution when k=2.

From (3) the Delta method provides an approximation which can be rewritten as:

$$\beta_1=\hat{\psi}\beta_2+b_2(\tilde{\psi}-\hat{\psi}) \tag{7}$$

this represents a line in the (β_1,β_2) space with a slope of $\hat{\psi}$ and an intercept given by $b_2(\tilde{\psi}-\hat{\psi})$. When the approximation is equal to the estimate $\tilde{\psi}=\hat{\psi}$ this line is a ray from the origin through the point defined by the estimates (b_1, b_2). By finding the two parallel lines that bound the ellipse defined by $(\boldsymbol{\beta}-\mathbf{B})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}-\mathbf{B}) = z^2$ we can determine the limiting values of $\tilde{\psi}$.

In Figure 1 we have drawn the constraint ellipse when $b_1 = 4.30$ and $b_2 = .66$, the estimate of the covariance is $\hat{\Sigma} = \begin{bmatrix} 3.02 & 0.05 \\ 0.05 & 0.23 \end{bmatrix}$ and $\alpha = .05$. The estimate of $\hat{\psi} = 6.51$ and the Delta standard error is estimated as 5.25 (using the expression for the variance given in (4)). These values imply that the 95% Delta confidence interval would be (16.8, -3.8).

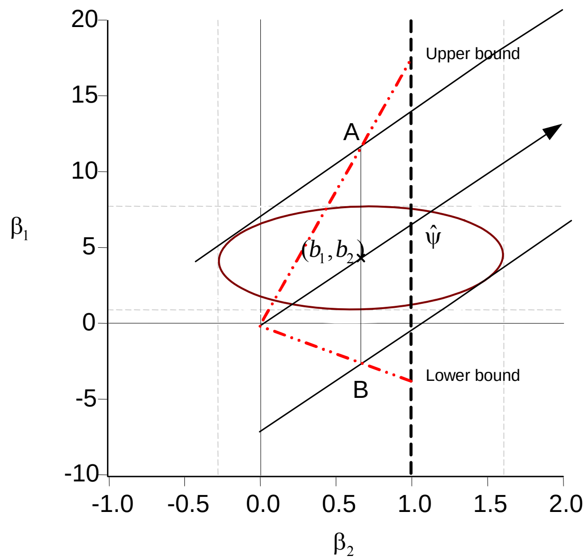

Figure 1 The construction of the Delta confidence interval for the ratio. Note that the bounds are defined by the rays from the origin through points A and B.

In order to read this result from Figure 1 we need to translate the intercept shift by the two limiting lines that are parallel to the estimated limit for the ratio. To do this we use the relationship $\beta_1 = \hat{\psi}\beta_2 + b_2(\tilde{\psi} - \hat{\psi})$ from which we find that the intercept is $b_2(\tilde{\psi} - \hat{\psi})$. We are interested in the change in $\tilde{\psi}$ and define these two intercept shifts as C_{up} and C_{low} (note $C_{low} = -C_{up}$). For the lower bound we can write $C_{low} = b_2(\tilde{\psi}_{low} - \hat{\psi})$ from which we can solve for $\tilde{\psi}_{low}$ as:

$$\tilde{\psi}_{low} = \frac{C_{low} + b_1}{b_2} \tag{8}$$

By adding the intercept shift either C_{up} or C_{low} to the estimate b_1 and dividing by b_2 we can find the bounds for the ratio. In Figure 1 these points are found on a line defined where the value of $\beta_2 = b_2$ as point A where $\beta_1 = b_1 + C_{up}$ and by point B where $\beta_1 = b_1 + C_{low}$. The two dotted lines from the origin pass through these points are for the two limiting ratios $\tilde{\psi}_{up}$ and $\tilde{\psi}_{low}$. We can read the values of the limiting ratios from the y-axis as the points where these rays from the origin intersect a line defined where $\beta_2 = 1$. In this case we can see that these limits are given approximately as 17 and -4. Also note that by construction these bounds are always symmetric and bounded for any possible ellipse shape or location.

We can summarize this process as follows:

1. Draw the ellipse defined by $\left[(b_1 - \beta_1)(b_2 - \beta_2)\right]\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}^{-1}\begin{bmatrix} (b_1 - \beta_1) \\ (b_2 - \beta_2) \end{bmatrix} = z^2$, where the elements of the covariance matrix are estimated by $\hat{\sigma}_{ij}$. Note the limits of this ellipse are the $(1-\alpha)$ 100% univariate bounds for β_1 and β_2. Here we continue to use the *z* statistic for simplicity though the equivalent *t* statistic could be used for small samples. Routines in a number of statistical software packages are available to accomplish this however it is necessary to make sure that the appropriate limiting value is used to insure this is the marginal ellipse (see Ruud 2000 page 230) and not the joint-test ellipse.

2. Construct the ray from the origin to the point (b_1, b_2) and the two bounding lines that are both parallel to this ray and tangent to the edges of the ellipse. These represent the bounding linear functions.

3. At the point (b_1, b_2) given by the estimates draw a line parallel to the y-axis (where $\beta_2 = b_2$) to find the points corresponding to A and B in Figure 1 where this line intersects the two bounding parallel lines drawn in step 2.

4. A ray from the origin through point A has a slope equal to the upper bound for ψ and the slope of the corresponding second ray from the origin through point B is the lower bound for ψ.

5. To read the estimates of the ratios from this diagram one find the points where the rays from the origin cut a line at $\beta_2 = 1$.

3. A geometric exposition of the Fieller interval

It can be shown that the Fieller interval for the ratio of two normally distributed random variables is equivalent to the solution to an optimization problem related to (6) where they both share the same constraint when *k* = 2. In this case the Lagrangian is defined by (see appendix B for details):

$$\mathcal{L} = \tfrac{\beta_1}{\beta_2} - \lambda\left(\left[(b_1 - \beta_1)(b_2 - \beta_2)\right]\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix}^{-1}\begin{bmatrix}(b_1 - \beta_1)\\ (b_2 - \beta_2)\end{bmatrix} - z^2\right) \tag{9}$$

Von Luxburg and Franz (2004) propose that a solution to this optimization problem can be found using a geometric representation. The ratio $\frac{b_1}{b_2}$ is the slope of the line through the points (0,0) and (b_1, b_2). In Figure 2 for the same bivariate normal distribution considered in Figure 1 we have plotted the ellipse and constructed the ray from the origin through (b_1, b_2). However in this case we define the Fieller limits by the rays from the origin that are tangent to the ellipse. The roots of the polynomial define the bounds of the ratio and when $\hat{\psi} > 0$ they are defined as:

$$\hat{\psi}_{1-\alpha/2}, \hat{\psi}_{\alpha/2} = \left(z^2\sigma_{22} - b_2^2\right)^{-1}\left(z^2\sigma_{12} - b_1 b_2 \pm z\begin{pmatrix}-z^2\sigma_{11}\sigma_{22} - 2b_1 b_2\sigma_{12}\\ +z^2\sigma_{12}^2 + b_2\sigma^2{}_{11} + b_1^2\sigma_{22}\end{pmatrix}^{½}\right) \tag{10}$$

Using the example considered above the Fieller lower bound is found to be 1.18. However, note that the upper bound as determined by ($\hat{\psi}_{1-\alpha/2}$) does not match the solution in that $\hat{\psi}_{1-\alpha/2} < \hat{\psi}_{\alpha/2}$. Therefore we conclude that the upper bound for this level of α is infinite. This can be found from an examination of Figure 2 where the upper bound would never cross the $\beta_2 = 1$ line. In this case the Fieller interval does not have a finite upper bound but does have a finite lower bound that, unlike the Delta lower bound, is greater than zero.

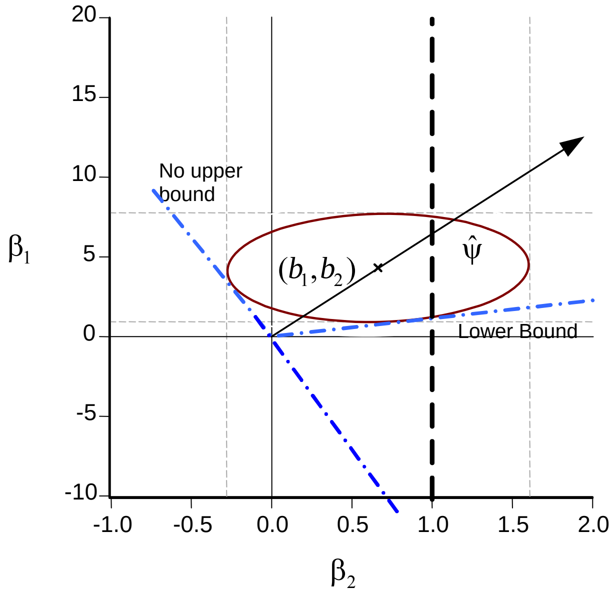

Figure 2 The geometric representation of the Fieller interval. Note that in this case there is no finite upper bound.

To summarize the process for constructing the geometric representation of the Fieller interval:

1. As in step #1 for the Delta, draw the ellipse defined by the constraint.

2. Construct the ray from the origin to the point (b_1, b_2) and then two bounding rays from the origin that are tangent to the edges of the ellipse. If the origin is interior to the ellipse then the bounds are infinite.

3. To read the estimates of the ratios from this diagram find where the rays from the origin cut a line defined by $\beta_2 = 1$. If the ellipse cuts the y-axis (as in the example shown in Figure 2) then one of the bounds is infinite (which bound depends on the sign of $\hat{\psi}$). When the ellipse cuts the y-axis this indicates that the $(1-\alpha)\,100\%$ univariate confidence interval for b_2 includes zero. When this is the case and $\hat{\psi} > 0$ then the upper bound is infinite and when $\hat{\psi} < 0$ then the lower bound is infinite.

4. The geometric comparison of different cases.

In this section we will provide some diagrams that include both the Fieller and the Delta intervals in order to demonstrate how the five estimated parameter values of $\beta_1, \beta_2, \sigma_{11}, \sigma_{22},$ and σ_{12} influence the relationship between the two intervals. In general, the greater the precision of the estimates the further the location of the ellipsoid from the origin. In addition, the sign of $\hat{\sigma}_{12}$ may have an impact on the differences between the two sets of confidence bounds by tilting the major axis of the ellipse in such a way as to maximize or minimize the differences. We consider the case where the ratio is equal to 1 or -1 to simplify the geometric exposition.

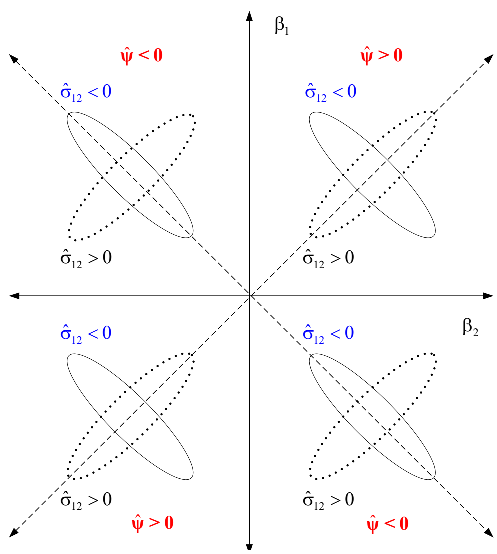

Figure 3 The relative positions of the ellipse when $\hat{\sigma}_{12} < 0$ (solid line) and $\hat{\sigma}_{12} > 0$ (dotted line) for different signs of the ratios.

In Figure 3 we present the possible cases for $\hat{\sigma}_{12} < 0$ and $\hat{\sigma}_{12} > 0$ for all combinations of the signs of the numerator and denominator when the ratio is either positive (in the NE and SW quadrants) or negative (in the SE and NW quadrants). When the sign of the estimated ratio is to equal the sign of the covariance $\left(\text{sign}(\hat{\sigma}_{12}) = \text{sign}(\hat{\psi})\right)$, the major axis of the ellipse and the ray from the origin are parallel to each other. In the alternative case, when $\left(\text{sign}(\hat{\sigma}_{12}) \neq \text{sign}(\hat{\psi})\right)$ the minor axis of the ellipse will be parallel to the ray from the origin and the major axis will be perpendicular to the ray from the origin.

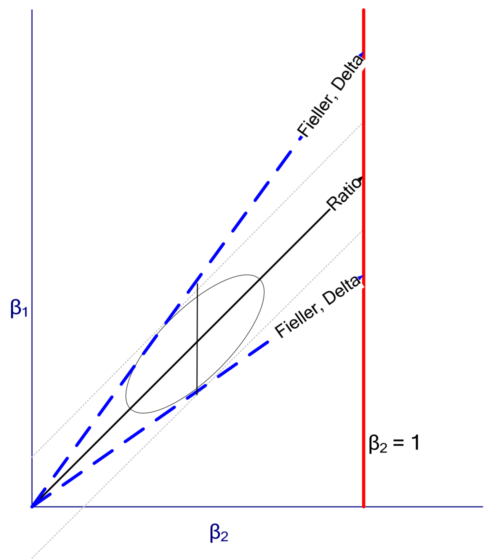

Figure 4 An example of the Fieller and Delta intervals when $\hat{\psi} > 0$ and $\hat{\sigma}_{12} > 0$. Note that the Fieller and the Delta intervals are almost identical.

In Figure 4 we demonstrate an example when $\hat{\psi} > 0$ and $\hat{\sigma}_{12} > 0$. In this case we find that the Delta and Fieller intervals are almost identical. In addition, this would be true even if the estimated parameters of the numerator and denominator (b_1, b_2) were smaller which would have the effect of moving the ellipse toward the origin.

Figure 5 shows the two bounds when the ellipse has the same location and variance as in Figure 4 except in this case the correlation is negative which results in a large difference in the Fieller and Delta bounds. The Fieller lower bound is higher and closer to the equivalent Delta case while the upper bound for the Fieller in this case is much greater than the corresponding bound for the Delta method. However in this case if the estimated parameters of the numerator and denominator (b_1, b_2) were greater, which would have the effect of moving the ellipse away from the origin, the differences between the Fieller and the Delta would diminish.

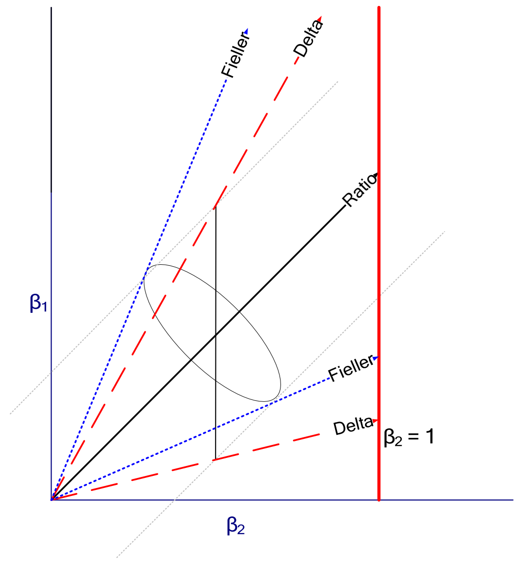

Figure 5 An example of the Fieller and Delta intervals when $\hat{\psi} > 0$ and $\hat{\sigma}_{12} < 0$. In this case the Fieller and the Delta intervals are quite different.

We can make some additional observations about the relationship between the orientation of the constraining ellipse and the degree to which the Delta coincides with the Fieller from the slope of the tangent to the ellipse at b_2. Thus we can establish if the ellipse is "lined up" with the ray from the origin that defines the estimate of the ratio. In Appendix C we derive the slope of the tangent to the ellipse at the value of the estimated denominator as $\hat{\rho}\sqrt{\hat{\sigma}_{11}/\hat{\sigma}_{22}}$, where $\hat{\rho}$ is the estimated correlation coefficient between the numerator and denominator. Thus when $b_1/b_2 = \hat{\rho}\sqrt{\hat{\sigma}_{11}/\hat{\sigma}_{22}}$, or equivalently, when the ratio of estimated *t*-statistics $\left(t_i = b_i/\sqrt{\hat{\sigma}_{ii}}\right)$ for the numerator and denominator equals the estimated correlation $t_1/t_2 = \hat{\rho}$. This condition was observed in a simulation study reported by Shanmugalingam (1982) as a case where distribution of the estimated ratio becomes approximately normal which provides the optimal conditions for the Delta approximation.

This relationship between the signs of the ratio and the correlation can also be found in Herson's (1975) analysis. He observed that "positive correlation is a more favorable situation for the approximation of a Fieller interval by a Delta Interval than a negative correlation" (page 268) from the computation of the relative difference of the 95% lower bound of the two intervals. In his Table 1

Herson describes a simulation in which he finds that when the ratio of coefficients of variation of the numerator to the denominator $\left(\frac{C_1}{C_2}\right)$, where $C_i = \hat{\sigma}_i / b_i$ or the inverse of the ratio of *t*-statistics is approaches the inverse of the correlation coefficient $\left(\frac{C_1}{C_2}\right) \to \hat{\rho}^{-1}$, the lower bound of the Delta most closely aligns with the Fieller.

5. Previous analytical comparisons of the Fieller and the Delta.

Finney (1952, page 63-4), (1978, page 80-82), Cox (1990) and Sitter and Wu (1993) among others present a method for imbedding the Delta interval into the Fieller interval. It can be shown that the Fieller interval can be defined as:

$$\hat{\psi}_{1-\alpha/2}, \hat{\psi}_{\alpha/2} = -\frac{1}{g-1}\left(\frac{b_1}{b_2} - g\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{22}} \pm \frac{z}{b_2}\left(\frac{1}{b_2^2\hat{\sigma}_{22}} \begin{pmatrix} b_2^2\hat{\sigma}_{11}\hat{\sigma}_{22} - 2b_1 b_2\hat{\sigma}_{12}\hat{\sigma}_{22} \\ -gb_2^2\hat{\sigma}_{11}\hat{\sigma}_{22} + b_1^2\hat{\sigma}_{22}^2 + gb_2^2\hat{\sigma}_{12}^2 \end{pmatrix} \right)^{1/2} \right) \tag{11}$$

where $g = z^2\hat{\sigma}_{22}/b_2^2$ - the ratio of the square of the critical value of $z_{(1-\alpha/2)}$ to the square of the estimated *t*-statistic value for the denominator $g = z^2/t_2^2$.

As $g \to 0$, the estimated absolute value of the *t*-statistic for the denominator becomes greater in magnitude (or equivalently the coefficient of variation becomes smaller), the Fieller interval becomes the same as the Delta interval defined in (5). In Figure 6 we show that the value of $\sqrt{g} = z\sqrt{\hat{\sigma}_{22}}/b_2$ is the distance from points A to B divided by the distance from A to O. Point A is the location of b_2, and point B is the $z\sqrt{\hat{\sigma}_{22}}$ limit of the univariate confidence bound for b_2.

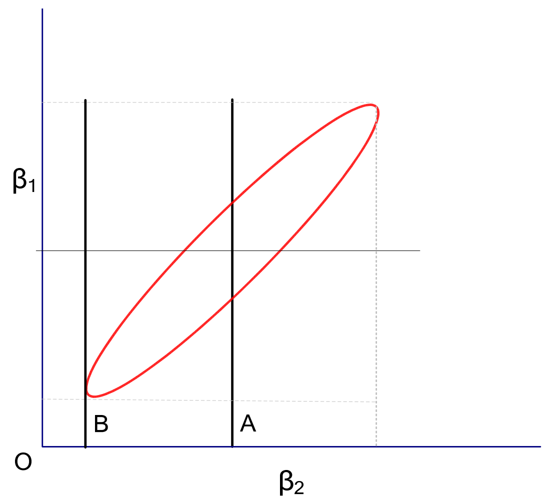

Figure 6 The value of $\sqrt{g} = z\sqrt{\hat{\sigma}_{22}}/b_2$ is the ratio of AB/AO.

Thus the further the ellipse from the origin the greater the value of g but this may only be part of the story, especially when the covariance is non-zero as we have shown in Section 4.

Finney (1978) suggests that a cut off value of $g^* = .05$ is sufficient to employ the Delta method as an approximation. If the critical value of z is 1.96, then the cut off value of $\sqrt{g^*} = .224$ implies a critical value of the estimated *t*-statistic for the denominator of 8.75 (or a coefficient of variation of .114). However, Finney (1978, page 82) cautions that in this calculation he has assumed a zero covariance between the numerator and denominator. Note that these values would imply that the ellipse would be located at a significant distance from the origin.

6. Previous simulation comparisons of the Fieller and the Delta.

Other comparisons of the Delta and the Fieller have been conducted in studies that compare the coverage of their implied confidence intervals. These studies use simulation experiments designed to mimic specific applications. In this section we review a sample of these results in light of our earlier findings in order to determine how the experimental design influenced the conclusions made in this literature. These investigations have been conducted in a variety of applications including: cost-effectiveness ratios, willingness to pay measures, bioassay applications for the median dose response, the validation ratio for surrogate endpoint evaluations, and long-run parameters from dynamic regression models.

The incremental cost-effectiveness ratio (ICER) is widely employed in the evaluation of proposed health treatment interventions. The numerator of the ICER is the incremental increase in costs and the denominator is the increase in effects and is thus expected to be positive. Polsky et al (1997) (table 2) find that the Delta intervals perform poorly for the case of negative correlations and best for positive correlations. In addition, when Briggs, Mooney and Wonderling (1999) analyse the results of a series of ICER simulations using a response surface regression they find that the correlation of the numerator to the denominator had the greatest positive impact on the coverage for the Delta method when compared to a number of alternative methods for interval construction.

Hole (2007) investigates the properties of the Delta for the willingness to pay measure defined as the ratio of parameters estimated in a logistic regression. In the application reported in table IX he concludes that the Delta and Fieller are very similar when the correlation was very small and the *t*-statistics for the denominator was quite large. His conclusion is not surprising considering that the constraint ellipse in this case is located quite far from the origin.

In Fieller's 1944 paper he defines a confidence interval for the median dose response level. These values are defined as ratios of estimated parameters and are assumed to be positive. Abdelbasit and Plackett (1983) find that the Delta and Fieller are the same using fairly large sample sizes of 243 and 447. This is a result that we would expect considering that large sample sizes imply smaller variances for means estimated from the same distribution and thus moving the ellipse further from the origin and resulting in smaller differences between the Fieller and the Delta intervals. Sitter and Wu (1993) investigate the properties of the Delta and the Fieller intervals using the same sample design but sample sizes in the range of 10 to 50. They find that the Delta and Fieller are different which is not surprising since such smaller sample sizes move the ellipse closer to the origin. Faraggi, Izikson and Reiser (2003) follow up this analysis with the recommendation that the Delta be avoided based on simulations where the correlations between the numerator and denominator are approximately -.4.

This result would coincide with our expectation that the Delta CI would not coincide with the Fieller CI when the sign of the estimated ratio does not agree with the sign of the correlation.

Freedman (2001) in a study of the use of the Delta and Fieller in the analysis of the *validation ratio* for *surrogate* or *intermediate* endpoints, as applied in epidemiological research, considers the confidence bounds of the ratio of regression parameters defined as: $\left(1-\beta_u/\beta_a\right)$. From his analysis he concludes that the Delta does not perform well due to the high positive correlation between β_a and β_u. This coincides with our conclusion that the Delta and Fieller intervals differ most when the signs of the ratio and the correlation are not the same (note that both β_a and β_u are assumed to have the same sign and to be close in magnitude).

Li and Maddala (1999) present an analysis of the Fieller as well as the Delta and various alternative resampling methods to the estimation of long-run elasticities from dynamic energy demand equations in which they report similar coverage for the Fieller and the Delta. By using the same data we found that the correlation between the numerator and denominator for the income elasticity measures were positively correlated which when matched with the positive sign of the estimated income elasticity for energy implies that the Delta method will be fairly accurate in this case.

7. Discussion

This paper has demonstrated that the geometric solution to two optimization problems can be combined to form a common representation of both the Delta and Fieller methods. The common geometric construction of these two intervals affords an opportunity to view how all the characteristics of the application under study influences the comparison of these methods.

Since its inception, a number of simulation studies have been performed to investigate the properties of the Fieller versus the Delta methods for the construction of confidence intervals for ratios of normally distributed random variables. The findings in this paper can be used to help design new experiments for particular applications so that the interrelationship between these tests will be best modelled for any particular class of applications. Specifically we have demonstrated the importance of the sign of the correlation between the numerator and denominator in conjunction with the sign of the ratio. If their correlation has the same sign as the estimated ratio the Delta and Fieller will coincide even when the denominator may be measured with a high variance. However, if they are negatively correlated they will diverge even in cases where the denominator has a fairly large *t*-statistic.

The methods used in this paper could be adapted to other nonlinear functions and the approximations that have been used to establish their confidence intervals. In addition, although they are not as tractable for geometric exposition, approximations used for higher dimensional functions could be examined in a similar manner. Another extension to this analysis is to the consideration of non-normal distributions for which the corresponding constraint may be other than an ellipse. In applications where the normal can not be assumed it may be possible to identify those features of the distribution that indicate the accuracy of approximations such as the Delta using an analytical or empirically constructed constraint.

References

Abdelbasit, K. M. and R. L. Plackett, 1983, "Experimental Design for Binary Data", *Journal of the American Statistical Association,* 78, 90-98.

Briggs, A. H., C. Z. Mooney and D. E. Wonderling, 1999, "Constructing Confidence Intervals for Cost-Effectiveness Ratios: An Evaluation of Parametric and Non-parametric Techniques using Monte Carlo Simulation", *Statistics in Medicine*, 18, 3245-3262.

Cox, C., 1990, "Fieller's Theorem, the Likelihood and the Delta Method", *Biometrics*, 46, 709-718.

Creasy, M. A., 1954, "Limits for the Ratio of Means", *Journal of the Royal Statistical Society, Series B*, 16, 186-194.

Durand, D., 1954, "Joint Confidence Regions for Multiple Regression Coefficients", *Journal of the American Statistical Association*, 49, 130-146.

Faraggi, D., Izikson, P. and Reiser, B., 2003, "Confidence Intervals for the 50% Response Dose", *Statistics in Medicine*, 22, 1977-1988.

Fieller, E. C., 1932, "The Distribution of the Index in a Normal Bivariate Population", *Biometrika*, 24,428-440.

Fieller, E. C., 1944, "A Fundamental Formula in the Statistics of Biological Assay, and Some Applications", *Quarterly Journal of Pharmacy and Pharmacology,* 17, 117-123.

Fieller, E. C., 1954,"Some Problems in Interval Estimation", *Journal of the Royal Statistical Society, Series B*, 16, 174-185.

Finney, D. J., 1952, *Probit Analysis: A Statistical Treatment of the Sigmoid Response Curve*, 2nd ed., Cambridge University Press, Cambridge.

Finney, D. J., 1978, *Statistical Method in Biological Assay*, 3rd ed., Charles Griffin & Company, London.

Freedman, L. S., 2001, "Confidence intervals and statistical power of the 'Validation' ratio for surrogate or intermediate endpoints", *Journal of Statistical Planning and Inference*, 96, 143-153.

Guiard, V., 1989, "Some remarks on the estimation of the ratio of the expected values of a two-dimensional normal random variable(correction of the theorem of Milliken)", *Biometrical Journal*, 31, 681-697.

Herson, J., 1975, "Fieller's Theorem vs. The Delta Methods for Significance Intervals for Ratios", *Journal of Statistical Computing and Simulation*, 3, 265-274.

Hinkley, D. V., 1969, "On the ratio of two correlated normal random variables", *Biometrika,* 56, 635-639.

Hirschberg, J. and J. Lye, 2007, "Providing Intuition to the Fieller Method with two Geometric Representations using STATA and EVIEWS", *Department of Economics*, *University of Melbourne, Working Paper* #992.

Hole, A. R., 2007, "A Comparison of Approaches to Estimating Confidence Intervals for Willingness to Pay Measures", *Health Economics*, 16, 827-840.

Li, H. and G. S. Maddala, 1999, "Bootstrap variance estimation of nonlinear functions of parameters: An application to long-run elasticities of energy demand", *The Review of Economics and Statistics*, 81, 728-733.

Marks, E., 1982, "A Note on a Geometric Interpretation of the Correlation Coefficient", *Journal of Education Statistics*, 7, 233-237.

Marsaglia, G., 1965, "Ratios of normal variables and ratios of sums of uniform variables", *Journal of the American Statistical Association*, 60, 193-204.

Polsky, D., H. A. Glick, R. Willke, and K. Schulman, 1997, "Confidence Intervals for Cost-Effectiveness Ratios: A Comparison of Four Methods", *Health Economics*, 6, 243-252.

Raftery, A. E. and T. Schweder, (1993), "Inference About the Ratio of Two Parameters, With Application to Whale Censusing", *The American Statistician*, 47, 259-264.

Ruud, P. A., 2000, *An Introduction to Classical Econometric Theory*, Oxford University Press, New York, NY.

Scheffé, H., 1959, *The Analysis of Variance*, John Wiley & Sons, New York, NY.

Shanmugalingam, S., 1982, "On the Analysis of the Ratio of Two Correlated Normal Variables", *The Statistician*, 31, 251-258.

Sitter, R. R. and C. F. J. Wu, 1993, "On the Accuracy of Fieller Intervals for Binary Response Data", *Journal of the American Statistical Association*, 88, 1021-1025.

Von Luxburg, U. and V. Franz, 2004, "Confidence Sets for Ratios: A Purely Geometric Approach to Fieller's Theorem", *Technical Report N0. TR-133*, Max Planck Institute for Biological Cybernetics.

Zerbe, G. O., 1978, "On Fieller's Theorem and the General Linear Model", *The American Statistician*, 32, 103-105.

Appendix A The equivalence of the constrained Optimization and the usual confidence intervals for linear combination of Normally distributed random variables.

Based on Scheffé (1959) appendix III we have constructed the following proof.

In the general case of a linear combination of a *k* dimensional normally distributed random vector:

$$\mathbf{B}_{k\times 1} \sim N\left(\boldsymbol{\beta}_{k\times 1}, \boldsymbol{\Sigma}_{k\times k}\right) \qquad \text{(A.1)}$$

$\boldsymbol{\Sigma}$ is assumed non-singular and the linear combination is defined as $\theta = \mathbf{a}'\boldsymbol{\beta}$ where $\mathbf{a}$ is a $k \times 1$ constant vector. We propose that the $(1-\alpha)100\%$ confidence interval for the estimate of θ $\left(\hat{\theta} = \mathbf{a}'\mathbf{B}\right)$ can be found from the solution to the constrained optimization defined by the Lagrangian defined as:

$$\mathcal{L} = \mathbf{a}'\boldsymbol{\beta} + \lambda\left(\left(\boldsymbol{\beta} - \mathbf{B}\right)' \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\beta} - \mathbf{B}\right) - z^2\right) \qquad \text{(A.2)}$$

Where z is the appropriate *z*-statistic for the $(1-\alpha)100\%$ confidence bound (i.e. for $\alpha = .05$ $z = 1.96$), and $z^2 = \chi_1^2$ the square of which is equivalent to a chi-square distributed random variable with one degree of freedom.

Taking the first derivatives of $\mathcal{L}$ with respect to the parameters and the multiplier and setting them equal to zero we find the following first order conditions which can be solved for the optimal values $\tilde{\boldsymbol{\beta}}$ and $\tilde{\lambda}$:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\beta}} = \mathbf{a} + \tilde{\lambda} 2\boldsymbol{\Sigma}^{-1}\left(\tilde{\boldsymbol{\beta}} - \mathbf{B}\right) = 0 \qquad \text{(A.3a)}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \left(\tilde{\boldsymbol{\beta}} - \mathbf{B}\right)' \boldsymbol{\Sigma}^{-1}\left(\tilde{\boldsymbol{\beta}} - \mathbf{B}\right) - z^2 = 0 \qquad \text{(A.3b}$$

Rewriting (A.3a) we find that:

$$\tilde{\boldsymbol{\beta}} - \mathbf{B} = -\tfrac{1}{2}\tilde{\lambda}^{-1}\boldsymbol{\Sigma}\mathbf{a} \qquad \text{(A.4)}$$

Which can then be substituted into (A.3b) to solve for $\tfrac{1}{4}\tilde{\lambda}^{-2}$:

$$\tfrac{1}{4}\tilde{\lambda}^{-2} = z^2\left(\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}\right)^{-1} \qquad \text{(A.5)}$$

By definition the covariance matrix is positive definite thus $\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a} > 0$ and we can find the square root of both sides of (A.5) to obtain: $\tfrac{1}{2}\tilde{\lambda}^{-1} = \pm z\left(\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}\right)^{-\frac{1}{2}}$. By subtracting **B** from both sides of (A.4) we find the optimal value of $\tilde{\boldsymbol{\beta}} = \mathbf{B} + \tfrac{1}{2}\tilde{\lambda}^{-1}\boldsymbol{\Sigma}\mathbf{a}$ after pre-multiplying both sides by $\mathbf{a}'$ and then substituting for $\tfrac{1}{2}\tilde{\lambda}^{-1}$ we find the constrained linear combination $\tilde{\theta} = \mathbf{a}'\tilde{\boldsymbol{\beta}}$ is defined as:

$$\tilde{\theta} = \mathbf{a}'\mathbf{B} \pm z\left(\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}\right)^{\frac{1}{2}} \qquad \text{(A.6)}$$

Which is the usual expression for the $(1-\alpha)100\%$ confidence interval of a linear combination of multivariate normally distributed random variables.

Appendix B The equivalence between the ratio restricted by the confidence ellipse and the Fieller Method.

Following the form of the discussion in Von Luxburg and Franz (2004) the bounds of the ratio of the means where the restriction is defined by the confidence ellipsoid of the two parameters can be found from the solution to the following constrained optimization problem:

$$\mathcal{L} = \tfrac{\beta_1}{\beta_2} - \lambda\left(\left[(b_1 - \beta_1) \quad (b_2 - \beta_2)\right]\begin{bmatrix}\omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22}\end{bmatrix}\begin{bmatrix}(b_1 - \beta_1) \\ (b_2 - \beta_2)\end{bmatrix} - z^2\right) \qquad \text{(B.1)}$$

where λ is the Lagrange multiplier, b_1 and b_2 are the estimated parameters, ω_{ij} are elements of the inverse of the covariance of the means, and z^2 is the square of the critical value of the normal distribution for a two tailed test.

Rewriting this Lagrangian in terms of ψ where $\psi = \beta_1/\beta_2$ and $\beta_1 = \beta_2\psi$:

$$\mathcal{L} = \psi + \lambda \begin{pmatrix} -z^2 + (b_2 - \beta_2)\left(\omega_{22}(b_2 - \beta_2) + \omega_{12}(b_1 - \psi\beta_2)\right) \\ + (b_1 - \psi\beta_2)\left(\omega_{12}(b_2 - \beta_2) + \omega_{11}(b_1 - \psi\beta_2)\right) \end{pmatrix} \tag{B.2}$$

The first order partial derivatives of $\mathcal{L}$ with respect to $\tilde{\beta}_2$, $\tilde{\psi}$, and $\tilde{\lambda}$ are defined as:

$$\tfrac{\partial \mathcal{L}}{\partial \psi} = 2\tilde{\lambda}\left(\tilde{\beta}_2^2\omega_{12} - b_2\tilde{\beta}_2\omega_{12} - b_1\tilde{\beta}_2\omega_{11} + \tilde{\psi}\tilde{\beta}_2^2\omega_{11}\right) + 1 \tag{B.3a}$$

$$\begin{aligned} \tfrac{\partial \mathcal{L}}{\partial \lambda} &= 2b_1b_2\omega_{12} + b_1^2\omega_{11} + b_2^2\omega_{22} - z^2 - \tilde{\beta}_2\left(2b_1\omega_{12} + 2b_2\omega_{22} - \tilde{\beta}_2\omega_{22}\right) \\ &\quad - 2\tilde{\psi}\tilde{\beta}_2\left(b_1\omega_{11} + b_2\omega_{12} - \tilde{\beta}_2\omega_{12}\right) + \tilde{\psi}^2\tilde{\beta}_2^2\omega_{11} \end{aligned} \tag{B.3b}$$

$$\tfrac{\partial \mathcal{L}}{\partial \beta_2} = 2\tilde{\lambda}\left(\beta_2\left(\omega_{22} + 2\tilde{\psi}\omega_{12} + \tilde{\psi}^2\omega_{11}\right) - b_2\omega_{22} - b_1\omega_{12} - \tilde{\psi}\left(b_1\omega_{11} + b_2\omega_{12}\right)\right) \tag{B.3c}$$

The first order conditions for an optimum are given by setting these partial derivatives to zero. First we can solve $\tfrac{\partial \mathcal{L}}{\partial \beta_2} = 0$ for $\tilde{\beta}_2$. Then we substitute for $\tilde{\beta}_2$ in in the equation for $\tfrac{\partial \mathcal{L}}{\partial \lambda} = 0$ which results in a quadratic equation in $\tilde{\psi}$. The roots of this quadratic are given by:

$$\tilde{\psi}_i = \frac{\left(-b_1b_2\omega_{11}\omega_{22} - z^2\omega_{12} + b_1b_2\omega_{12}^2 \pm z\begin{pmatrix} z^2\left(\omega_{12}^2 - \omega_{11}\omega_{22}\right) + 2b_1b_2\omega_{11}\omega_{12}\omega_{22} + b_2^2\omega_{11}\omega_{22}^2 \\ -2b_1b_2\omega_{12}^3 - b_1^2\omega_{11}\omega_{12}^2 + b_1^2\omega_{11}^2\omega_{22} - b_2^2\omega_{12}^2\omega_{22} \end{pmatrix}^{1/2}\right)}{z^2\omega_{11} - b_2^2\omega_{11}\omega_{22} + b_2^2\omega_{12}^2} \tag{B.4}$$

Alternatively, the Fieller method is defined as the solution for the values of $\tilde{\psi}$ is given as in (10):

$$\tilde{\psi}_i = \left(z^2\sigma_{22} - b_2^2\right)^{-1}\left(z^2\sigma_{12} - b_1b_2 \pm z\begin{pmatrix} -z^2\sigma_{11}\sigma_{22} - 2b_1b_2\sigma_{12} \\ +z^2\sigma_{12}^2 + b_2\sigma^2{}_{11} + b_1^2\sigma_{22} \end{pmatrix}^{1/2}\right) \tag{B.5}$$

Using the correspondence between the covariance matrix and its inverse defined as:

$$\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} = \left(\sigma_{12}^2 - \sigma_{11}\sigma_{22}\right)^{-1}\begin{bmatrix} -\sigma_{22} & \sigma_{12} \\ \sigma_{12} & -\sigma_{11} \end{bmatrix} \tag{B.6}$$

We can show that the roots for the constrained optimization problem solution in (B.4) are equal to the expression (B.5).

Appendix C The determination of the slope of the constraint ellipse evaluated at the estimated values.

Following Marks (1982) we derive the slope of tangents to the constraint ellipse. Redefining the constraint ellipse in terms of the correlation coefficient $\rho = \sigma_{12}\big/\sqrt{\sigma_{11}\sigma_{22}}$ we have:

$$0 = \begin{bmatrix} (b_1 - \beta_1) & (b_2 - \beta_2) \end{bmatrix}\begin{bmatrix} \sigma_{11} & \rho\sigma_{11}\sigma_{22} \\ \rho\sigma_{11}\sigma_{22} & \sigma_{22} \end{bmatrix}^{-1}\begin{bmatrix} (b_1 - \beta_1) \\ (b_2 - \beta_2) \end{bmatrix} - z^2 \tag{C.1}$$

where b_1 and b_2 are the estimated parameters, and z^2 is the square of the critical value of the normal-distribution for a two tailed test. Taking the partial derivative of β_1 with respect to the value of β_2 we obtain:

$$\tfrac{\partial \beta_1}{\partial \beta_2} = \sqrt{\frac{\sigma_{11}}{\sigma_{22}}}\left(\rho \pm (b_2 - \beta_2)\sqrt{\frac{\rho^2 - 1}{b_2^2 - 2b_2\beta_2 + \beta_2^2 - t^2\sigma_{22}}}\right) \tag{C.2}$$

When evaluated at the estimate $\beta_2 = b_2$ we obtain: $\tfrac{\partial \beta_1}{\partial \beta_2} = \rho\sqrt{\tfrac{\sigma_{11}}{\sigma_{22}}}$.